Epistemology

Universals and Measurement

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I. Orientation

Rand (1966–67, 1, 13) spoke of universals as abstractions that are concepts. Quine ([1961] 1980, 70) spoke in the same vein of "conceptual integration—the integrating of particulars into a universal." Those uses of *universal* engage one standard meaning of the term. Another standard meaning is the potential collection to which a concept refers. This is the collection of class members consisting of all the instances falling under the concept.[1] In the present study, the character of universals in the latter sense will be brought into fuller articulation and relief. That vantage will be attained by amplifying the mathematical-metaphysical requirements of Rand's theory of universals as conceptual abstractions.

To begin, I situate the topic of the present study within Rand's larger system of metaphysics and epistemology. My core task for the present study will then emerge fully specified.

Rand's system relies on three propositions taken as axioms: (E) Existence exists; (I) Existence is identity; (C) Consciousness is identification.[2] Rand's set of axioms conveys the fundamental dependence of consciousness on existence. Existence is and is as it is independently of consciousness, whereas consciousness is dependent on existence and characters of existence (Rand 1957, 1015–16; 1966–67, 29, 55–59; 1969–71, 228, 240–41, 249–50).

As part of the meaning of (I), Rand contended (I_m): All concretes, whether physical or mental, have measurable relations to other concretes (1966–67, 7–8, 29–33, 39; 1969–71, 139–40, 189, 199–200, 277–79).[3] Every concrete thing—whether an entity, attribute, relation, event, motion, locomotion, action, or activity of consciousness—is measurable (Rand 1966–67, 7, 11–17, 25, 29–33; 1969–71, 184–87,

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223–25).

As part of the meaning of (C), Rand made the point (C_m): Cognitive systems are measurement systems (1966–67, 11–15, 21–24; 1969–71, 140–41, 223–25). Perceptual systems measure,[4] and the conceptual faculty measures. Concepts can be analyzed, according to Rand's theory, as a suspension of particular measure values of possible concretes falling under the concept. Items falling under a concept share some same characteristic(s) in variable particular measure or degree. The items in that concept class possess that classing characteristic in *some* measurable degree, but may possess that characteristic in *any* degree within a range of measure delimiting the class (Rand 1966–67, 11–12, 25, 31–32).[5] This is Rand's "measurements-omitted" theory of concepts and concept classes.

All concretes can be placed within some concept class(es). All concretes can be placed under concepts. Supposing those concepts are of the Randian form, then all concretes must stand in some magnitude relation(s) such that conceptual rendition of them is possible. What is the minimal magnitude structure (minimal ordered relational structure) that all concretes must have for them to be susceptible to being comprehended conceptually under Rand's measurement-omission formula?

That is to say, what magnitude structure is implied for metaphysics, for all existence, by the measurement-omission theory of concepts in Rand's epistemology? My core task in the present study is to find and articulate that minimal mathematical structure. With that structure in hand, we shall have as well the fuller articulation of the class character of universals implied by Rand's theory of concepts.

Such mathematical structure obtaining in all concrete reality is metaphysical structure. It is structure beyond logical structure; constraint on possibility beyond logical constraint. Yet it is structure ranging as widely as logical structure through all the sciences and common experience.

The minimum measurement and suspension powers required of the conceptual faculty by Rand's theory of concepts call for neuronal computational implementation. Is such implementation possible, plausible, actual? This is a topic for the future, bounty beyond the present study.

We must keep perfectly distinct our theoretical analysis of concepts and universals on the one hand and our theory of the developmental genesis of concepts on the other. Analytical questions will be treated in the next section, and it is there that I shall discharge the core task for this study.

The logicomathematical analysis of concepts characterizes concepts per se. It characterizes concepts and universals at any stage of our conceptual development, somewhat as time-like geodesics of space-time characterize planetary orbits about the sun throughout their history. The analysis of concepts and universals offered in the next section constrains the theory of conceptual development, as exhibited in Section III.

II. Analysis

Rand gave three definitions of *concept*. I shall tie them all together in the next section, but for the present section, we need this one alone: Concepts are mental integrations of "two or more units possessing the same distinguishing characteristic(s), with their particular measurements omitted" (13).[6]

The *units* spoken of in this definition are items appropriately construed as units by the conceiving mind. They are items construed as units in two senses, as *substitution units* and as *measure values* (Rand 1969–71, 184, 186–88). As substitution units, the items in the concept class are regarded as indifferently interchangeable, all of them standing as members of the class and as instances of the concept. Applied to concept units in their substitution sense, measurement omission means release of the particular identities of the class members so they may be treated indifferently for further conceptual cognitive purposes.[7] This is the same indifference at work in the order-indifference principle of counting. The number of items in a collection may be ascertained by counting them in any order. Comprehension of counting and count number requires comprehension of that indifference.

The release of particular identity for making items into concept-class substitution units is a constant and necessary part of Rand's measurement-omission recipe. But this part is not peculiar to Rand's scheme. What is novel in Rand's theory is the idea that in the release of particular identity, the release of which-particular-one, there is also a suspension of particular measure values along a common dimension.

Before entering argumentation for the minimal mathematical structure implied for the metaphysical structure of the world, let us check that we have our proper bearings on objective structure and intrinsic structure. I have ten fingers, eight spaces between those fingers, and two of my fingers are thumbs. That's how many I have of those items. Period. Those numerosities are out there in the world, ready to be counted, and they are what they are whether I count them or not. In our positional notation for expressing and calculating numbers, we choose the number base, but the different base systems designate the same things, the numbers. In base ten, my (fingers, spaces, thumbs) are (10, 8, 2); in base eight (12, 10, 2); and in base two (1010, 1000, 10). The three numbers referred to in all these bases are the same three numbers. In Rand's terminology, the various bases are objective schemes; they are appropriate tools for getting to the intrinsic structure of numbers. But the numbers have intrinsic character—even or odd, whole or fraction, rational or irrational, analytic or transcendental—quite independently of our choices, such as choice of number base.

In asking for the minimal magnitude structure that all concretes must possess if all concretes can be subsumed under concepts for which Rand's measurement-omission analysis holds, we are seeking intrinsic structure, obtaining under every adequate objective expression of that structure. Now we are ready.

1. Affordance of Ratio or Interval Measures

I have said that the units suspended in the formula "two or more units possessing the same distinguishing characteristic(s), with their particular measurements omitted" are units in a double sense: substitution units and measure values. We focus now on units in the latter sense. Rand (1966–67, 39) spoke of measurement as "identification of a relationship in numerical terms" and as "identification of a relationship—a quantitative relationship established by means of a standard that serves as a unit" (7; also 33; see further 1969–71, 188, 199–200). The measure-value sense of *unit* is the one at work here. By the expression "a standard that serves as a unit" and by some of her examples of concepts and their measurement bases, one might suppose that Rand's theory of concepts entails that all concretes stand under magnitude relations affording some sort of concatenation measurement. That supposition would be incorrect.

Rand illustrates her theory with the concept *length*. The pertinent magnitudes of items possessing length are magnitudes of spatial extent in one dimension. Another illustration of Rand's is the concept *shape* (1966–67, 11–14; 1969–71, 184–87). The pertinent magnitudes of items possessing shape, in 3D space, are pairs of linear, spatial magnitudes such as curvature and torsion for shapes of curves or the two principal curvatures for shapes of surfaces.[8]

Shapes must possess such pairs of magnitudes in *some* measure but may possess them in *any* measure. Observe that Rand's measurement-omission theory does not entail what number of dimensions for the magnitude relations among concretes is appropriate for the concept. *Length* requires 1D, *shape* requires 2D. Rand's theory works for any dimensionality and does not entail what the dimensionality must be, except to say that it must be at least 1D. Observe also that the conception of linearity to be applied here to each dimension is not the more particular linearity familiar from analytic coordinate geometry or from abstract vector spaces. It is merely the linearity of a linear order.[9]

The magnitude structure of the concretes falling under the concept *length* affords concatenations. Take as unit of length a sixteenth of an inch. Copies of this unit can be placed end-to-end, in principle, to form any greater length, such as foot, mile, or light-year. This standard concatenation of lengths is properly represented mathematically by simple addition. That is a numerical rule of combination appropriate to concatenations of the concrete magnitude structure in the case of length.

The magnitude structure of the concretes falling under the concept *length* also affords ratios that are independent of our choice of elementary unit. The ratio of the span of my left hand, thumb-to-pinky, to my height is simply the number it is, regardless of whether we make those two measurements using sixteenths of an inch as elementary unit or millimeters as elementary unit.

Mass is another concept whose concept-class magnitude structure affords simple-addition concatenations and affords ratios of its values that are independent of choice of elementary unit. Because of the latter feature, conversion of pounds to kilograms requires only multiplication by a constant. Such measurement scales are called ratio scales.[10] The mathematical combinations reflecting the concatenations need not be simple addition. This category of scales is somewhat more inclusive than that. It would include the scale for the concept *grade* (grades of roads, say). Grades can be concatenated, although the proper mathematical reflection of this concatenation is not simple addition.[11]

Finest objectivity requires measurement scales appropriate to the magnitude structures to which they are applied. What does *appropriate* mean in this context? It means that all of the mathematical structure of the measurement scale is needed to capture the concept-class magnitude structure of concretes under consideration. It means as well that all the magnitude structure pertinent to the concept class is describable in terms of the mathematical structure of the measurement scale.[12]

What is the magnitude structure of concretes that is appropriately reflected by ratio-scale characterization? It is a magnitude structure whose automorphisms are translations.[13] Translations are transformations of value-points (i.e., points, which may be assigned numerical values) of the magnitude structure (the ordered relational structure of the concept-class concretes) that shift them all by the same amount, altering no intervals between them.

Rand's measurement-omission analysis of concepts and concept classes applies perfectly well to cases in which the measurement scale appropriate to the pertinent magnitude structure of concretes is ratio scale. But Rand's theory does not entail that all concretes afford ratio-scale measures. For Rand's theory does not necessitate that the scale type from which measurements be omitted be ratio scale. Her analysis also works perfectly well for scales having less structure. The magnitude structure entailed for all concretes by Rand's theory is less than the considerable structure that ratio scales reflect.

An analogous conclusion obtains for multidimensional magnitude structures of concept classes. Rand's theory does not entail that all 2D or 3D magnitude structures have both affine structure and absolute structure, as Euclidean geometry has.[14] That is, Rand's theory does not entail that multidimensional magnitude structures of concept classes afford a metric (a measure of the interval between two value-points) definable for all pairs of points from a scalar product (a measure of perpendicularity of value-lines).[15]

Physical temperature, certain aspects of sensory qualities, and certain aspects of utility rankings are examples of concretes whose magnitude structures afford what are now called interval measures, but evidently do not afford ratio measures.[16] The magnitude structure underlying the concept class *temperature* affords only an interval scale of measure. Such magnitude structures do not afford concatenations, unlike the natures of length or mass, but they do afford ordering of differences of degree, and they afford composition of adjacent difference-intervals.[17]

Such magnitude structures do not afford ratios of degrees that are independent of choice of unit, but they afford ratios of difference-intervals that are independent of choice of unit and choice of zero-point.[18] Ratio scales have one free parameter, requiring we select the unit, such as yard or meter. These scales are said to be 1-point unique. Interval scales have two free parameters, requiring we select the unit, such as °F or °C, and requiring we select the zero-point, such as the freezing point of an equally portioned mixture of salt and ice or the freezing point of pure ice. These scales are said to be 2-point unique.[19]

The magnitude structure of concretes affording interval-scale characterization is one whose automorphisms are fixed-point collineations, preeminently stretches.[20] Stretches are transformations of the value-points of a magnitude structure such that one point remains fixed and the intervals from that point to all others are altered by a single ratio.

Rand's measurement-omission analysis of concepts and concept classes applies perfectly well to cases in which the measurement scale appropriate to the pertinent magnitude structure of concretes is interval scale. The temperature attribute of a solid or fluid must exist in *some* measure, but may exist in *any* measure.[21] But Rand's theory does not entail that all concretes afford interval-scale measures. For Rand's theory does not necessitate that the scale type from which measurements be omitted be interval scale. Her analysis also works perfectly well for a kind of scale having less structure. The magnitude structure entailed for all concretes by Rand's theory is still less than the considerable structure that interval scales reflect.

An analogous conclusion obtains for multidimensional magnitude structures of concept classes. Rand's theory does not entail that all 2D or 3D magnitude structures have not only order structure, but affine structure, as Euclidean and Minkowskian geometry have.[22] That is, Rand's theory does not entail that multidimensional magnitude structures of concept classes afford a metric definable from a norm (a measure on vector structure).[23]

2. Affordance of Ordinal Measures

Recall again Rand's characterization of measurement: identification of "a quantitative relationship established by means of a standard that serves as a unit" (1966–67, 7). The phrase "a standard that serves as a unit" suggests that Rand's conception of measurement for her measurement-omission analysis of concepts was ratio-scale or interval-scale measurement. These two types possess interval units that can serve as interval standards. They possess interval units that can be meaningfully summed to make measurements. The quantitative relationship established in measurements equipped with interval units entails summation of elementary units. The summation might be simple addition or a more elaborate mathematical combination, and the basis for the summation in concrete reality might be susceptibility to concatenation (for ratio scales) or to composition of ordered difference-intervals (for interval scales).

The measure values required for Rand's theory need not be interval units. As Rand realized, merely ordinal measurement suffices for her measurement-omission scheme (33). I say that the magnitude structure captured by ordinal measurement is the minimal structure implied for metaphysics if, as I supposed at the outset, all concretes fall under one or more concepts for which Rand's measurement-omission analysis holds. What is the magnitude structure captured by ordinal measurements?

All magnitude structures captured by ratio- or interval-scale measurements contain a linear order relation. A magnitude structure consisting only of such a linear order relation is a structure for which merely ordinal measurement is appropriate. An example is the *hardness* of a solid. I mean specifically the scratch-hardness, which is measurable using the Mohs hardness scale. Calcite scratches gypsum, but not vice versa; quartz scratches calcite, but not vice versa; therefore, yes, quartz scratches gypsum, but not vice versa. Degrees of hardness have an order that is anti-symmetric and transitive.

Mohs scale assigns the numbers (2, 3, 8) to the degrees of hardness for (gypsum, calcite, quartz). All that is intended by the scale is to be true to the order of the degrees of hardness. That Mohs has chosen these three numbers to be integers is of no significance. They could as well be the rational triple (14.7, 55.3, 56.9) or the irrational triple ($\sqrt{2}$, π, 1.1π). Unlike the numbers on interval scales, the ratios of difference-intervals between the numbers on these scales are not meant to be of any significance. The hardness degrees (2, 3, 8) are not intended to imply that the hardness of calcite is closer to the hardness of gypsum than it is to the hardness of quartz. For all we know, and for all our ordinal measurements signify, there simply may be no fact of the matter whether the scratch-hardness of calcite is closer to that of gypsum than to that of quartz.

The magnitude structure of hardness (scratch-hardness, not dent-hardness) evidently does not warrant summations or equal subdivisions of some sort of interval unit of hardness. This particular *hardness* concept is founded analytically on merely ordinal measure. To fall under this concept *hardness*, an occasion need only present the quality at *some* measure value on the merely ordinal scale, and that may be *any* measure value on that scale. Affordance of ordinal measurement is all that Rand's measurement-omission recipe entails for the magnitude structure of all concretes. Her theory does not entail that every attribute of concretes—hardness, for example—must in principle afford ratio- or interval-scale measurements. Her theory does not imply that, were only our knowledge improved enough, it would be possible to make ratio- or interval-scale measurements of scratch-hardness.[24]

The magnitude structure affording merely ordinal measurement is a linear order whose automorphisms are the order-automorphisms of (same-order subsets of) the real numbers in their natural order. Such a magnitude structure affords characterization by a lattice (a type of partially ordered set) formed of sets and subsets of possible Dedekind-cuts of its linear order. This linear order might be scattered or dense; ordinal measurement is possible in either case.[25]

The magnitude structure affording merely ordinal-scale measurement affords metrics. Each of the three scales adduced above to capture degrees of hardness bears a metric defined by the absolute values of those scales' numerical differences. A magnitude structure affording a (separable) metric belongs to the topological category known as a (separable) uniformity. Topologies that are uniformities in this sense are Hausdorff topologies, but they need not be compact nor (topologically) connected.[26] The topological character of the magnitude structure entailed for all concretes by Rand's measurement-omission theory of concepts is the character of a uniformity.

The magnitude structure entailed by Rand's theory has the algebraic character of a lattice, which has more structure than a partially ordered set (or a directed set) and less than a group (or a semi-group). In terms of the mathematical categories, Rand's magnitude structure for metaphysics is a hybrid of two: the algebraic category of a lattice and the topological category of a uniformity. Rand's structure belongs to the hybrid we should designate as a uniform topological lattice.

Concerning multidimensional magnitude structures of concept classes, I concluded in the preceding subsection that Rand's theory entails neither affine nor absolute structure. What *is* entailed: concept classes with a 2D or 3D magnitude structure will have the structure of at least an ordered, distance geometry.[27] Significantly, it is implied that planes and spaces concretely realizable will have at least that much structure.

3. Superordinates and Similarity Classes

Hardness, fatigue cycle limit, critical buckling stress, shear and bulk moduli, and tensile strength all fall under the superordinate concept *strength* of a solid. The conceptual common denominator (Rand 1966–67, 15, 22–25; 1969–71, 143–45) of these various strengths of solids is that they are all forms of resistance to degradations under stresses. What is the common measure of this resistance the different species of strength have in common? What is the common measure of strength that all specific forms of the strength of solids have in common? The magnitude structure of hardness affords only ordinal-scale measurement. The magnitude structure of tensile strength affords ratio-scale measurement. Only the ordinal aspect of the tensile-strength measure could be common with the hardness measure. Is the ordinal aspect of each specific form of strength a same, single, common measure? No, the ordinal measure of hardness is not the same as the ordinal measure of tensile strength. Resistance to being scratched is not the same as resistance to being pulled apart under tensile stress.

The way in which an ordinal measure of hardness and an ordinal measure of tensile strength can form a common ordinal measure for the general concept *strength* of a solid is only as substitution units, not as distinct measure values along some common ordinal measurement scale. The *some-any* locution can be applied to substitution units (e.g., Rand 1966–67, 25). We sensibly say that strength of a solid in general must have *some* type of ordinal strength measure but may have *any* such type. That sort of use of *some-any* pertains to units as substitution units: there must be *some* specific form of strength to instantiate the general concept *strength* of a solid, but it may be *any* of the specific forms.

The substitution-unit standing of concepts under their superordinate concepts is a constant and necessary part of Rand's measurement-omission recipe as applied to the superordinate-subordinate relationship. But this part is not peculiar to Rand's scheme for that relationship. Here is what is novel in Rand's measurement-omission theory for superordinate constitution, as I have dissected it: Whichever concept is considered as an instance of the superordinate concept, not only will that subordinate concept and *its* instances stand as a substitution instance of the superordinate, each instance of the subordinate will have some particular measure value along a specific dimension. And that particular value is suspended for the subordinate concept, thence for the superordinate concept.

Analytically, identity precedes similarity.[28] For purposes of her theory of concepts and concept classes, Rand defined similarity as "the relationship between two or more existents which possess the same characteristic(s), but in different measure or degree" (13). I concur. Occasions of scratch-hardness are similar to each other because they are all occasions of scratch-hardness, exhibiting that hardness in various measurable degrees. This much accords with Rand's definition and use of similarity in the theory of concepts.

Occasions of scratch-hardness are also more like each other than they are like occasions of tensile strength. This is a perfectly idle invocation of comparative similarity (comparative likeness). The work that comparative similarity pretends to be doing here can be accomplished fully by simple identity (sameness) without any help from similarity: scratch-hardness is itself and not something else, such as tensile strength.

The shapes of balls are similar to each other because they have principal-curvature measures at various values within certain ranges. Likewise for the shapes of cups (to keep the illustration simple, consider a Chinese teacup, not a cup with a handle). Moreover, ball shapes are more like one another than they are like cup shapes because ball values of principal curvatures are closer to each other than they are to cup values of principal curvatures. Here the invocation of comparative similarity is not idle. To say that ball shapes are more like one another than they are like cup shapes is to say something beyond what is claimed in saying: Shapes that balls have are themselves and not something else, such as shapes that cups have.

The strengths of a solid are of various kinds that are not simply of various values along some common dimension(s). The shapes of a solid are of various kinds, and unlike kinds of strengths, these kinds are of various values along some common dimension(s).

Rand's conception of similarity as sameness of some characteristic, but difference in measure, can be put squarely to work in analyzing comparative similarities of shapes of solids with each other. Then this conception of similarity is a genuine worker, too, in the analysis of the concept *shape* of a solid, superordinate for the concepts *ball-shape* and *cup-shape*. This employment of Rand's conception of similarity in the analysis of comparative similarity, thence in the analysis of superordinates, is just as Rand would have it (14). But such an employment of Rand's conception of similarity as sameness of some characteristic, but difference in measure, is incorrect in application to the comparative similarities of the various strengths of solids, thence to their superordinate concept *strength* of a solid.

What will be the proper analysis as we move on up the superordinates? Strengths of a solid are more like strengths of a solid than they are like shapes of a solid. Let us suppose, as Rand supposed, that the reason we can say that strengths of a solid are more like each other than they are like shapes of a solid is because there is some common dimension, the dimension of the conceptual common denominator, between strengths and shapes of a solid. *Property* of a solid fits the bill for conceptual common denominator. Strengths and shapes of a solid are both properties of a solid. What is the measurable dimension of the concept *property* of a solid that is common to both *strength* and *shape* of a solid? Like the common dimension for strength, it is a dimension consisting of nothing more than various substitution dimensions. The measurable dimension of *property* of a solid will be the hardness dimension or the tensile-strength dimension or the principal-curvature dimensions or . . . There is no single, common measure of *property* of a solid that all specific properties of solids have in common. Rand supposed in error that there were, for she supposed it always the case that there is some same, common measurable dimension supporting the conceptual common denominator for any superordinate concept (23).[29] That supposition is here rejected, and measurement-omission analysis of superordinate concepts is here corrected in this respect.

Suppose for a moment, though it be false, that there were some common measurable dimension of *property* of a solid that was singular, not common merely by substitutions. Then in saying that strengths of a solid are more like each other than they are like shapes of a solid, we could reasonably contend that the values of strengths are closer to each other on the hypothetical common *property*-of-a-solid dimension than they are to the values of shapes on that common dimension (14).[30] Then the magnitude structure of the common dimension for *property* of a solid could not be one that affords only ordinal measures. On such measures, there is no telling whether a value between two others is closer to the one than to the other. (Then in an order of values ABCD, one has no measure-basis for clustering B or C with A or D: B might cluster with A, and C might cluster with D; or B and C might both cluster with A; or . . .) The magnitude structure of the common dimension for *property* of a solid would have to afford additional measurement structure. It would need to afford ratio- or interval-scale measurements. But it is not at all plausible that a measurable dimension common to each instance of *property* of a solid should have not only ordinal-scale structure, but ratio- or interval-scale structure, when *hardness*, for instance, has only ordinal structure.

4. Amended Measure-Definitions of Similarity and Concepts

With possible exception for the most general concepts such as *property*, Rand supposed that concept classes are always similarity classes (1969–71, 275–76). This is immediately apparent from comparison of her definition of similarity with her definition of concepts. In the present study, I likewise make Rand's supposition.

Now I have said that a solid's resistance to being scratched is not the same as its resistance to being pulled apart under tensile stress. Nonetheless, these two sorts of strength of a solid are similar. Occasions of hardness are similar to occasions of tensile strength because the same characteristic, limit of resistance to some sort of stress, is possessed by both in different measurable forms. These measurable forms could be merely ordinal, yet in this way be a basis of similarity. Moreover, hardness and tensile strength are more similar to each other than to shape because hardness and tensile strength are two different measurable forms of the same characteristic that is different from the measurable characteristic (pair of principal curvatures) shared by shapes in different degrees.

So I should amend Rand's definition of similarity as follows: Similarity is the relationship between two or more existents possessing the same characteristic(s), but in different measurable degree or in different measurable form.

The corresponding definition of concepts would be: Concepts are mental integrations of two or more units possessing the same distinguishing characteristic(s), with their particular measurements omitted or with the particular measurable forms of their common distinguishing characteristic(s) omitted.

Every concrete falls under both sorts of concept. Both sorts of conceptual description have application to every concrete. Occasions of hardness fall under the *hardness* concept by sameness of characteristic in various measures. Those very same occasions of hardness fall under the *strength* concept by sameness of characteristic varying in measurable form. Occasions of cup-shape fall under the *cup-shape* concept by sameness of (pairs of) characteristics in various measures. Those very same occasions of cup-shape fall under the *spatial property* concept by sameness of a characteristic, spatial extension, that has various measurable forms.

5. Conclusion of Core Task

My amendments to Rand's definitions of concept and concept class (similarity class) do not implicate metaphysical structure beyond what is already implied by Rand's definitions. Where I have spoken of various measurable forms of a characteristic, all of those forms are the same measurable dimensions that are also at work in concept classes based on variation of measure values along a dimension.

What is the magnitude relation under which all concretes must stand such that conceptual rendition of them is possible? They must stand in the relation of a uniform topological lattice, at least one-dimensional. This is the magnitude structure implied for metaphysics, for all existence, by the theory of concepts in Rand's epistemology. The same magnitude structure is implied by that theory with my friendly amendment.

What is the mathematical character of universals, of the collection of potential concept-class members, implicit in Rand's theory of concepts? In Rand's theory, universals are recurrences, repeatable ways that things are or might be. Properties, such as having shape or having hardness, are examples of such ways. That universals are recurrences is a traditional and current mainstay in the theory of universals.[31] In Rand's theory, however, universals are not only recurrences, they are recurrences susceptible to placement on a linear order or they are superordinate-subordinate organizations of recurrences susceptible to placement on such linear orders.

Universals as (abstractions that are) concepts are concept classes with their linear measure values omitted. If the concept is a superordinate, then the linear measurable form might also be omitted, that is, be allowed to vary across acceptable forms. Universals as collections of potential concept-class members are recurrences on a linear order with their measurement values in place.[32] For either sense of the term *universals*, they are an objective relation between an identifying subject and particulars spanned by those universals (Rand 1966–67, 7, 29–30, 53-54; 1965, 18; 1957, 1041).

III. Genesis

Rand takes concepts to be mental products of a mental process "that integrates and organizes the evidence provided by man's senses" ([1970] 1982, 90). She gives three definitions of concepts:

(1) Concepts are mental integrations of "two or more perceptual concretes, which are isolated by a process of *abstraction* and united by means of a specific definition" ([1961] 1964, 20);

(2) More generally in terms of the data processed, concepts are mental integrations of "two or more units which are isolated according to a specific characteristic(s) and united by a specific definition" (1966–67, 10);

(3) Finally and most deeply, concepts are mental integrations of "two or more units possessing the same distinguishing characteristic(s), with their particular measurements omitted" (12).

The "two or more perceptual concretes" spoken of in definition (1) are the elementary type of "two or more units" spoken of in (2) and (3). Rand proposes, in a general way, a developmental intellectual ascent from apprehending the world only in terms of perceptual concretes and actions they afford to apprehending that same world in terms of units in classes. That ascent is a refinement and sophistication in our apprehensions of existents: an ascent from apprehending existents as entities to apprehending them as identities to apprehending them as units (1966–67, 6–7; 1969–71, 180–81).

Rand's measurement-omission analysis of concepts could be correct even if her account of their genesis were incorrect. In particular, her analysis could be correct even if her proposed developmental intellectual ascent were incorrect. I contend that her general proposed ascent is correct. I shall give a thumbnail sketch of the developments I think should be seen as tracing an entity-identity-unit ascent in the apprehension of existents.

1. Elaboration of Identity

For the first day or two after birth, existents for us are plausibly only entities. Such would be the occasions of Mother's face or voice.[33] Very soon existents become for us not mere entities, but identities, particular and specific.[34]

At 20 *days*, there is expectation of the reappearance of a visual object gradually occluded by a moving screen; here is rudimentary particular identity of visual objects (Bremner 1994). At 4 *weeks*, there is some oral tactile-to-visual transfer of object features, without opportunity for associative learning; here is rudimentary specific identity (Meltzoff 1993). At 5 weeks, there is recognition memory of color and form; here is growth of specific identity. At 8 weeks, there is onset of attention toward internal features of patterns and onset of smooth visual tracking, and there is hand tactile-to-visual transfer of object features; here is growth of particular and specific identity. At 10 weeks, there is expectation that one visual solid object cannot move through another (Bremner 1994). By 3 *months*, visual tracking is becoming anticipatory (Johnson 1990); there is visual fill-in of invisible parts of objects (Bremner 1994); visual objects are being identified as separate using various static-separation and motion traits (Spelke and Van de Walle 1993); there is categorical perception of objects and events (Quinn 1987). At 4 months, haptic apprehensions of shapes can be transferred to the visual mode (Streri and Spelke 1988); visual solid objects are expected to endure and retain size when occluded for a brief period (Bremner 1994); objects are expected to fall if not supported (Needham and Baillargeon 1993).

The infant's world of entities-identities will continue to elaborate. Units are not yet. At 6 months, the infant will have some sensitivity to numerosity; will be able to detect numerical correspondences between disparate collections of items, even correspondences between visible objects and audible events; and will be able to detect the equivalence or nonequivalence of numerical magnitudes of collections (Starkey, Spelke, and Gelman 1990). At 7 months, still without words, the infant distinguishes global categories (e.g., animals v. vehicles), which will later become superordinates of so-called basic-level categories (e.g., dog v. car) yet to be formed (Mandler and Bauer 1988; *cf.* Rand 1969–71, 213–15). By 12 months, the infant reliably interprets adult pointing, looking from hand to target (Butterworth and Grover 1988).

2. First Words, First Universals

At around 12 months, the infant puts first words, single-word utterances, into her play. Words at this stage are used only in play, not for communication, which is still accomplished with cries, gestures, and gazes (Bremner 1994, 249–51; Nelson 1996, 105, 112). An infant in my family, just past his first birthday, uses the word *ba*. He says it quietly to himself whenever he sees or is handed a spherical ball; he does not say his word when the ball is a football. We should not suppose too hastily, I should note, that his word *ba* refers simply to the spherical ball with which he is engaged. At this first-words stage, his utterance may designate the object as component of his whole activities that go with those objects (activities like training the adults to fetch) (Bremner 1994, 251–52; Nelson 1996, 97, 109–10, 115, 227–29; Bloom 2000, 35–39).

By 14 months, the toddler points to indicate items (Butterworth and Grover 1988). By 16 months, she *spontaneously* groups objects of a single category (Bremner 1994, 173). In another month or two, comes the naming explosion, naming of objects especially (Nelson 1996, 111–15; Macnamara 1986, 144–45; Bloom 2000, 91–100). (That there really is such a dramatic burst in the rate of word acquisition at this time is disputed; Bloom 2000, 39–43.)

By that time, at 17 or 18 months, the toddler is using single words to refer (Macnamara 1986, 56–57). These words (50 to 100 words) include demonstratives such as *that*, common nouns such as *ball*, and proper names such as *Star*, say, to refer to a particular ball. The use of common nouns and proper names in single-word reference indicates certain competencies of identification, certain representational comprehensions of identities specific and particular. The representational comprehensions of specific and particular identity that are evidently coming into operation at this stage are class-membership relation, individuation within a class, and particular identity over time.

Skillful reference for the utterance *ball* indicates that the beginning speaker has some working principles for deciding whether a given item qualifies as being in the category *ball*. Then such a speaker has some operational sense of class-membership relation (61–62, 72–74, 124–28, 148–49, 152–56). *Ball* is a count noun. Although the beginning speaker does not yet possess the principles of counting, not even implicitly, she has some working principles of individuation within a class, some principles for holding in mind individual balls as distinct from one another (128–30).[35] Moreover, *ball* refers to any individual ball as a distinct individual over time (59–60, 130–36, 141–42, 152). Finally, the name *Star* is attached to a particular one of those individual balls over time (55–62, 71–83).[36]

I suggest that even at the single-words stage of language development, the toddler has entered the conceptual level of consciousness in Rand's sense of that level. The utterance *ball* refers, and marks a concept, already at this stage.

One problem for that conjecture is the following. Rand required that the items falling under a concept be united with a specific definition. [See (1) and (2) above and Rand 1966–67, 48–50; 1969–71, 177–81.] But at the single-words stage of development, the toddler cannot yet form two-word expressions. That competence will not be attained for another six months or so, at around 24 months of age.[37] Not yet having two-word expressions, she cannot yet form a sentence, cannot yet use words in assertive sentences. Without propositions one is without defining propositions, hence, without definitions. Then at the single-words stage of development, the items falling under a "concept" cannot be united by a specific definition.

Then it would seem one does not yet possess a concept in Rand's sense. I think that conclusion would be an overstatement.

For an older child or an adult, of course, "a concept identifying perceptual concretes stands for some implicit propositions" (Rand 1966–67, 48, 21). For a single-words toddler, no propositions can be adduced. Actions can be adduced. A *ball* is something that can be handled and thrown down. It bounces and rolls. These things are clearly known of balls even by the one-year-old whose first and only word is *ba*. The concept *ball* is likely held in mind in the form of image and action schemata as well as by the term *ball* (1966–67, 13, 20, 43; 1969–71, 167–70).[38]

There is something else, something profoundly conceptual, at hand in linguistic competence at least by the time of the naming explosion. John Macnamara (1986, 124–28) concludes that having a word such as *ball* at this stage means having a logical principle of application. That is a surrogate for definition at this single-words stage. A principle of application is the working principle, spoken of above, for determining whether an item is or is not a ball.[39] A principle of application determines class membership. That is the basic function a definition accomplishes for more advanced language users (Rand 1966–67, 40; 1969–71, 231–32).

To have an operational grasp of the class-membership relation is to have a tacit grasp of the notion of unit in the sense of a substitution unit, which is the unit for counting. That does not mean that one has yet grasped the elementary principles of counting (nor that one can put the notion of a substitution unit to work in counting). At 18 months, one has evidently gotten some working hold on the notion of a substitution unit, the notion of a simple member-of-a-class, without yet having the principles of counting. But at this single-words stage, one has taken the first step into the dual realms of the conceptual and the mathematical. With a tacit grasp of the notion of unit in the substitutional sense, "man reaches the conceptual level of cognition, which consists of two interrelated fields: the Conceptual and the Mathematical" (Rand 1966–67, 7). Rand was correct in thinking that "man's mathematical and conceptual abilities develop simultaneously," even though she was incorrect in thinking that "a child learns to count when he is learning his first words" (1966–67, 9; 1969–71, 200).[40]

Having *ball*, one is getting hold of "ball, any one." That is the membership relation and its requisite principle of application. Having *ball*, one is also getting hold of "some things of a class, the balls" (Rand 1966–67, 17–18). That is the individuation-within-class relation and principle (Macnamara 1986, 128–30). Then already at the stage of first concepts, one has beginning working principles of universal quantification (*any*) and existential quantification (*some*).[41]

3. Analytic Constraint

As we have seen, in Rand's view, in the analysis of any concept there can be found a double application for *some* and *any*: with respect to substitution units and with respect to measure values. To form our concepts, however, Rand supposes that we do not need to grasp, expressly or tacitly, the notion of units as measure values. We discern similarities. Where there is similarity, there can be found various measure values along a common dimension, in Rand's view, but we need not know anything about such measure bases.

When we pick up a ball, our sensory systems measure it in several ways. When we perceive a similarity between two items, according to Rand's account, we are perceiving some same characteristic(s) they both possess in different measure or degree (1966–67, 13–14; 1969–71, 139–40, 143). They both possess that characteristic in some measure or degree. Items of their class possess that characteristic in *some* degree, but may possess it in *any* degree within a range of measure delimiting the class (Rand 1966–67, 11–12, 25, 31–32).

On which characteristic(s) does the similarity class, thence the concept class, rest? Like Ockham, Rand observed that items in a similarity class are more similar to (and less different from) one another than they are to things not in the class. A ball is more similar in various ways to other balls than it is to sticks, hands, and so forth. As we know, Rand analyzed similarity in terms of measurable dimensions, in terms of measures of dimensional characteristics. The characteristic(s) on which the similarity class and its concept *ball* rests, analytically and genetically, in Rand's theory, is whichever measurable characteristic(s) makes a ball measurably closer to other balls than to sticks, hands, and so forth (1966–67, 13–14, 21–23, 41–42; 1969–71, 144–47, 217, 274–76).[42]

I have addressed the defect and remedy of this measure-theoretic analysis of similarity classes and concepts in the preceding section. It remains to address the genetic aspect, which I cast as: in forming a similarity class and its concept, one is relying on (tacitly using) whichever measurable characteristic(s) makes items in that class and under that concept measurably closer to one another than to opponent items.

Rand thought, rightly I should say, that formation of any concept whatever requires differentiating two or more existents from other existents. She thought also that such differentiation requires comparative degrees of difference, measurable as such on a dimension(s) common to existents in the class and existents outside the class (Rand 1966–67, 13). What if Rand were right in this second doctrine? What if, in order to form any concept whatever, there had to be a dimension common to the concept class and its opponents and this had to be a dimension along which comparative closeness measurement is possible? What would that imply for metaphysics? It would imply that every concrete can be placed in concept classes whose linear measures are not only ordinal-scale, but interval- or ratio-scale as well.[43]

I avoid that extravagant implication as follows: I retain Rand's assumption that formation of any concept requires differentiating two or more existents from other existents and her assumption that all concept classes are similarity classes and her measure-definitions, as amended above, of concepts and similarity. I reject the assumption that differentiation between existents included in and existents excluded from a concept class require comparative degrees of difference (beyond the comparative-difference-degree pretender that merely says a thing is less different from itself than it is different from things not itself).

Such differentiation may sometimes be based at least partly on fairly blunt sameness and difference. Spherical balls are the same with one another in that they roll regularly, and in this they are baldly different from floors. A dimension along which items in a concept class have various measure values need not be a dimension common with items in an opponent concept class.

Differentiation of existents included in or excluded from a concept class *may* enlist nontrivial comparative degrees of difference (or likeness). I see three forms of these. In one, the comparative degrees are along dimensions common to both included and excluded existents, and those dimensions afford either ratio- or interval-scale measures. Along the dimensions of shape, a spherical ball can be distinguished from a football in that way. The sets of pairs of principal curvatures (ratio scaling) over the surfaces of spherical balls are less different from each other, from one ball's set of pairs to another ball's set of pairs, than they are from the sets of pairs of principal curvatures over the surfaces of footballs.

In light of my amendment to the measure-definition of similarity, we should allow also for a second nontrivial variety of comparative similarity. I observed earlier that hardness and tensile strength are two different measurable forms of a same characteristic (resistance to degradation under some sort of stress) that is different from the measurable characteristic (pairs of principal curvatures, which are spatial extension properties) shared by shapes. This second manner of decomposing a comparative similarity permits concepts based on comparative degrees of similarity without requiring that linear measures of the concept dimensions be anything beyond ordinal measures.

A third decomposition of nontrivial comparative similarity does not rely on shared and unshared dimensions of the relata. It relies simply on numbers of shared and unshared features.[44] Perhaps any concept based on this sort of comparative similarity can be recaptured in a more sophisticated way by ascertaining measurable dimensions on which to base the concept (Boydstun 1990, 31–33). I expect that is so. Notice, however, that the metaphysical implication drawn in the present study (uniform topological lattice structure) need not suppose that all concepts can be analyzed in terms of Rand's measurement-omission formula; only that all concretes can be placed under one or more concepts analyzable in those terms.

I have exhibited a way in which a measurement analysis of concepts can constrain theorizing about the genesis of concepts. I do not want to create the impression, however, that theory of the genesis of concepts based on observations and empirical testing cannot rightly constrain one's analysis of concepts. The analytical principles stating that all concretes can be placed in concept classes having a measurement structure and that these structures are of such-and-such characters are conjectures open to restriction through counterexamples. These conjectures of analysis are subject to reform or replacement in the face of contrary analytical and empirical results, somewhat as the General Relativity principle that freely falling bodies follow time-like geodesics of space-time is subject to reform or replacement.[45]

One of the avenues for empirical confrontation of our analytical conjectures concerning concepts and concept classes is research on conceptual development. The core task I have undertaken in the present study has been a certain extension of Rand's metaphysics arising from her analysis of concepts (not her theory of their formation). I have not undertaken here a survey of the various ways in which empirical research on conceptual development may challenge Rand's analysis of concepts. But there is one form of challenge that is invalid, and I want to draw attention to this fallacy, which has required a long struggle for me to overcome. That is a fallacy I insinuated in Boydstun 1990, 33–34. It says that because preschoolers do not possess—not even tacitly—mathematical understanding sufficient to be forming their concepts using a principle of measurement-omission, their concepts do not bear analysis in terms of measurement-omission. That is the fallacy of confusing genesis with analysis.[46]

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Notes

1. *Cf.* Armstrong 1978a, 25–26.

2. Rand takes propositions (E), (I), and (C) to express primary facts and to be fundamental compositions upon three concepts she takes as axiomatic: *existence*, *identity*, and *consciousness*. She takes all concepts to bear implicit propositions that elucidate the concepts (Rand 1966–67, 48; 1969–71, 177–81, 228). Propositions (E), (I), and (C) are immediate elucidations of Rand's axiomatic

concepts (1957, 1015–16). Rand does not present (I) and (C) as axioms, only as most important elucidations of her three axiomatic concepts; for her order of presentation, she follows what she takes to be the order of cognitive development (1966–67, 3, 55–56, 59). My order of presentation brings the propositions (E), (I), and (C) to the fore, and this, I hope, is analytically illuminating.

3. In the case of the concrete that is the universe itself, which is all of existence, the measurable relations are to parts of itself. For example, the total mass-energy of the universe is a measure having relation to each of its constituents having mass-energy.

Rand took (I_m) to be axiomatic in that she took it to be entailed by her axiom (I). A thing not measurable in any way "would bear no relationship of any kind to the rest of the universe, it would not affect nor be affected by anything else in any manner whatever, . . . in short, it would not exist" (1966–67, 39). Rand is supposing that anything bearing some relationship to the rest of the universe bears some measurable relationship to the rest of the universe. I think that this supposition, which is tantamount to (I_m) (all concretes have measurable relations to other concretes), is a postulate additional to the axiomatic postulate (I) (existence is identity). I do not regard the postulate (I_m) to be axiomatic; unlike the axiom (I), the postulate (I_m) can be denied without self-contradiction and is therefore open to possible restriction by counterexamples. Like Rand, I take (I_m) to be an unrestrictedly true postulate.

4. This idea is widespread. Antecedents are to be found in Schopenhauer, Goethe, Helmholtz, James, Bergson, and Dewey. For current applications of the idea that perceptual systems measure, see Krantz, Luce, Suppes, and Tversky 1989, 131–53; Churchland and Sejnowski 1992, 183–233.

5. Pale anticipations of this idea of Rand's may be found in James [1890] 1950, 270, Heath [1925] 1956, 132–33, Johnson [1921] 1964, 173–92, Prior 1949, and Searle 1959. For relations to Aquinas and Hume, see Boydstun 1990, 24–27.

6. How might the concept *existence* satisfy this definition of concepts? Might the concept of existence as *all existents* (Rand 1969–71, 241) be rendered as all instances of existents, with all measure values of those existents omitted (1966–67, 56)? See also Armstrong 1997, 194–95.

7. A concept class is at least the sense of class at work for a kind, for mere membership in a kind (Macnamara 1986, 50–53, 152–56). For Rand's theory of concepts, however, it seems that concept classes might always also be properly regarded as sets. For in Rand's theory, all concept classes must be measurable. They must afford some appropriate numerical representation, and any such representation can also be expressed in terms of sets.

There are reasons to doubt whether concept classes always satisfy even the extensionality postulate of Zermelo-Fraenkel set theory, the postulate that two classes collecting the same items are the same class (Macnamara 1986, 152; Bigelow 1988, 102). Concept classes not satisfying that postulate could not qualify as either so-called proper classes or as sets. Even if those doubts can be put to rest (Bigelow 1988, 101–9), there would remain further doubts about whether absolutely all concept classes satisfy the separation axiom of ZF set theory. Some concepts, such as the concept *all items* (all things that are either a *potential or actual existent* or a mere *posit*), are so comprehensive that they do not themselves stand as substitution units in some superordinate concept. Then concept classes need not always be extensionality-satisfying classes that are also sets. In particular, a concept class need not always be itself a member of a larger class. Such concepts are extremely rare; almost always an extensionality-satisfying concept class will qualify as a set. I assume, with trepidation, that concept classes appropriate for

Rand's theory of concepts are not only classes in the sense of a kind, but also are rightly construed as classes that satisfy extensionality and, with rare exception, are rightly construed as classes that are sets (*cf.* Armstrong 1997, 185–95).

The following are proper classes, extensional classes that are not sets: the class of all items (the universe of class discourse), the class of all sets, the class of all ordinals (all order types of total orders having least members), and the class of all cardinals (all least ordinals for sets such that there is a mapping from least order type to set that is one-to-one and onto). On proper and nonproper classes, see Machover 1996, 10–16; von Neumann [1925] 1967, 393–94, 403; Quine [1961] 1980, 90–101, 112–17; 1982, 94, 130–31, 302; and Boolos 1998, 35–36, 42–47, 73–87, 223–24, 238–41.

8. See Kline 1972, 554–64, 882–86; Churchland and Sejnowski 1992, 183. My measurement analysis of the concept *shape* supplements Rand's. Likewise, it could supplement Armstrong's (see Armstrong 1997, 55–56).

9. *Cf.* Rand 1966–67, 14; Peikoff 1991, 84. On the intended elementary sense of linear order, see Rosenstein 1982, 3.

10. On ratio scales, see Krantz, Luce, Suppes, and Tversky 1971, 3–5, 9–10, 44–46, 71–87, 518; 1990, 10, 108–13.

11. Simple addition: 153.0 yards joined to 153.0 yards is 306.0 yards. Grade addition: 153.0 yards per mile joined to 153.0 yards per mile is 310.3 yards per mile. Note also that there are valid and specific nonstandard ways of concatenating lengths, and these are faithfully represented mathematically by specific nonsimple additions (Krantz, Luce, Suppes, and Tversky 1971, 87–88, 99–102; 1990, 18–56).

12. *Cf.* Swoyer 1987, 256–58. Here I take a norm accepted in mathematical physics and adapt it for our broader context (Geroch 1985, 86, 81–84, 119). *Physical* gets replaced by *concrete* for our metaphysics. Notice, making that replacement, that to obtain the relation of mathematics to metaphysics, we may look to the relation of mathematics to physics (1, 17, 111–13, 183–87, 223, 283–90, 324–40; see also Geroch 1996).

13. See Krantz, Luce, Suppes, and Tversky 1990, 112–25; Martin 1982, 14–17. Structures are characterized by their automorphisms, the set of structure-preserving morphisms of that structure into itself. (Consider the set of rotations and reflections, confined to the plane, that transform a square into itself: 90° rotation about the square's center, reflection through a diagonal line, and so forth.) The identity morphism, taking every point into itself, is among the set of automorphisms for any structure. The set of automorphisms for a totally disorganized structure has only that one member, the identity morphism. The identity automorphism by itself affords counting, which belongs to the type of measurement known as absolute measurement (Suppes 2002, 110–18). That barest structure is less than the minimal structure required for concept classes under Rand's measurement-omission analysis of concepts.

14. Contrast Rand's system in this respect with the systems of Descartes and Kant. Rand's theory does not entail that there is *any* 2D or 3D magnitude structure of concretes having the structure of Euclidean geometry. In particular, Rand's theory does not imply that physical space is Euclidean.

Let me also note at least some of what is meant by *absolute* and *affine* in the present context. Euclidean geometry contains both absolute geometry and an affine geometry. Absolute geometry consists of those propositions of Euclidean geometry that can be obtained from Euclid's first four postulates alone, neither affirming nor denying the fifth postulate, which is the parallel postulate. These propositions hold not only in Euclidean geometry but in hyperbolic geometry.

Absolute structure permits the comparison of lengths along lines whether or not they are parallel to each other.

Affine geometry consists of those propositions that can be obtained from Euclid's first two postulates (to draw a straight line from any point to any point and to produce a finite straight line continuously in a straight line) together with the fifth postulate (in one version: for any point P off a line L, there exists a unique line through P that is parallel to L). Affine structure permits the comparison of lengths only along lines that are parallel to each other. See Krantz, Luce, Suppes, and Tversky 1989, 109–11; Coxeter 1980; Martin [1975] 1998.

15. See Krantz, Luce, Suppes, and Tversky 1989, 31–35.

16. It might be thought that temperature was found to afford ratio scaling once absolute zero was conceived and the "absolute thermodynamic temperature scale" was constructed. That is incorrect. The interval units of the absolute thermodynamic temperature scale (°K) are the same as the interval units of the Celsius scale (°C). Like the Celsius and Fahrenheit scales, construction of the absolute thermodynamic temperature scale requires not only that an interval unit be chosen, but that a fundamental fixed point be chosen and assigned a value. The fixed point selected for the absolute thermodynamic temperature scale is the triple point of water (unique temperature and pressure at which water, ice, and vapor coexist). Absolute zero is then *defined* to be 273.16 °K below the triple point exactly.

What if, contrary to my supposition, temperature were found to be a physical quantity that affords ratio measures? That would not change the outcome of my core task in this study. I am to delineate and put aside the richer types of magnitude structures affording measurement until we arrive at the minimal structure required for Rand's measurement-omission recipe. The physical examples presented need be, for our purpose, only hypothetical illustrations of types of magnitude structures.

On applications of interval-scale measurement in psychophysics, see Krantz, Luce, Suppes, and Tversky 1971, 139, 519–20; 1989, 177–78, 184–85; also, Michell 1999, 20–21, 74–76, 81–87, 147–52, 172–77, 189–90, 198–200, 205–8. On applications of interval-scale measurement in utility theory, see Krantz, Luce, Suppes, and Tversky 1971, 17–21, 139–42; also Nozick 1985. That there are magnitude structures affording interval-scale measurement in the realms of utility and psychophysics does not mean that every magnitude structure in those realms affords such measures; some may afford merely ordinal measurement. Rand (1966-67, 32–34) likely supposed that only ordinal measurement is appropriate in utility theory, under tutelage from Austrian-school economists (*cf.* Rothbard 1962, 15–28, 222–31, 276–79).

17. On interval scales, see Krantz, Luce, Suppes, and Tversky 1971, 10, 17–21, 136–48, 170–73, 515–20; 1990, 10, 108–13. Throughout this paper, I use simply *concatenation* in place of the usual technical expression *positive concatenation*. That the concatenations are positive means that the resulting, concatenated magnitude is greater than either of the magnitudes entering into the concatenation. So, I say simply that magnitude structures of concretes such as temperature (or chemical potential) do not afford concatenations, rather than say, as would be usual technically, that such structures afford concatenations qualified as *intensive* in contrast to *positive*.

18. A body or fluid at 43°C is at 109.4°F. If at 45°C, then at 113.0°F. If at 56°C, then at 132.8°F. The Celsius difference-interval ratio (45 – 43)/(56 – 45) equals the Fahrenheit difference-interval ratio (113.0 – 109.4)/(132.8 – 113.0). The simple ratios of degrees such as 43/45 and 109.4/113.0 are not equal, unlike

the character of ratio scales. We should be aware too of an important respect in which magnitude structures affording interval scales are like magnitude structures affording ratio scales. For either type of structure and their scale types, it is the case that whether two intervals in the structure are equal is independent of which measurement scale in the scale type is used. The interval between 43°C and 45°C equals the interval between 47°C and 49°C. That equality remains when those values are converted to °F, though the value of each equal interval changes from 2°C to 3.6°F.

19. Ratio scales stand to each other as a metal bar under uniform thermal expansions. A single number characterizes a particular state of expansion, a particular ratio scale. Interval scales stand to each other as an elastic band pinned at some point, then stretched to some degree from that pinned point, for various such pinnings and stretches. Two numbers characterize a particular pinning and stretch, a particular interval scale. On characterization of scale types by degrees of uniqueness and homogeneity, see Krantz, Luce, Suppes, and Tversky 1990, 112–25, 142–50; Suck 2000; Cameron 1989.

20. Krantz, Luce, Suppes, and Tversky 1990, 112–22; Martin 1982, 136–44.

21. Temperature attributes are relational attributes, specifically, difference attributes. When we sense the warmth or coolness of a body by touching it, we are sensing the rate of heat flow into or out of our own body at the contact surface. Rate of heat flow reflects size of temperature difference between the two bodies in contact.

22. Krantz, Luce, Suppes, and Tversky 1989, 107–8; Coxeter 1980; Martin [1975] 1998.

23. See Krantz, Luce, Suppes, and Tversky 1989, 42–46. The concept *color* (resolved as hue, saturation, and brightness) is a 3D magnitude structure that is affine, but not also absolute (Krantz, Luce, Suppes, and Tversky 1971, 515–20; 1989, 40, 243–50, 279–85). The concept *space-time* (flat space-time) is a 4D magnitude structure that is affine, but not also absolute (though it has absolute substructures).

24. Rand did not herself reach a stable understanding of these entailments. See Rand 1966–67, 31; 1969–71, 189–90, where she expresses her supposition that our resort to measurements less rich than ratio-scale measurement is a resort to measurements that are less "exact" and reflects our relative ignorance of the thing we are measuring.

25. On linear orders, see Rosenstein 1982. On ordinal measurement, see Krantz, Luce, Suppes, and Tversky 1971, 2–3, 11, 14–15, 38–43; 1989, 83–89, and Droste 1987a; 1987b.

26. The absolute value function here is not taken over the real numbers in their character as a vector space. Then the absolute value function in our merely ordinal context is not a norm (Bartle 1976, 54–55). Our metric is not being derived from a norm; we do not magically convert our merely ordinal scale to an interval one by taking absolute values of numerical differences.

On topological, uniform, and metric spaces, see James 1999 and Geroch 1985. That the topology of a magnitude structure affording ordinal-, interval-, or ratio-scale measurement be a Hausdorff topology seems fitting. In such a topology, any two distinct points have some nonintersecting neighborhoods, and this would seem to be a natural condition for any sort of measurement at all.

27. On ordered geometry, see Krantz, Luce, Suppes, and Tversky 1989, 104–7, and Coxeter 1980. I say a *distance* geometry rather than a *metric* geometry because the distance function need be only positive and symmetric. The triangle inequality, an additional requirement for a metric, need not be satisfied (Martin

[1975] 1998, 68–69; Coxeter 1980, 175–81; Blumenthal 1970, 16; consider also, Krantz, Luce, Suppes, and Tversky 1989, 186–87, 205–8).

A mathematically determinate form from which measure values may be suspended for the concept *shape* of a curve (in 3D) is a set of curvature and torsion values, one pair of values for each point of the curve. Consider a 2D graph in which curvature values are plotted along one axis and torsion values are plotted along the other axis. Plotting the particular pairs of values for a particular curve in concrete 3D space will form a particular curve in the plane of our 2D graph. Relations among points in this plane satisfy the axioms of a 2D ordered, distance geometry (as well as axioms for richer 2D geometries).

The concept class *shape* of a curve satisfies my principle, sprung from Rand's measurement-omission theory of concepts, that all concept classes having a multidimensional magnitude structure have the structure of at least an ordered, distance geometry. Many of our concepts are obviously multidimensional. Consider a general-purpose definition of the concept *animal* (*metazoa*): a multicellular living being capable of nervous sensation and muscular locomotion. Surely the mathematically determinate form of the concept class *animal* is multidimensional (*cf.* Rand 1966–67, 16, 24–25, 42). My principle alleges that that multidimensional structure will have the structure of at least an ordered, distance geometry.

28. *Cf.* Armstrong 1978a, 44–50; 1978b, 95–123; 1997, 17–18, 22–23, 47–57; Jetton 1998, 41–42.

29. But consider Rand 1969–71, 275–76.

30. See also Rand 1969–71, 139–40, as well as Peikoff 1991, 85, and Gotthelf 2000, 59. Further, see Kelley and Krueger 1984, 52–61; Kelley 1984, 336–45; Jetton 1998, 63–72.

31. See Armstrong 1978a, 11–12, 77–87, 108–16; 1997, 14–15, 28–31, 49; Bigelow 1988, 4, 18–27, 40–41, 56–57, 121–78.

32. *Cf.* Armstrong 1997, 185–95. 33. Rand (1966–67, 5, 6) concluded from research literature as of 1966 that the sensory experience of the infant was apparently entirely "an undifferentiated chaos" and did not contain any percepts. Subsequent research has dispelled that old vision of cognition in newborn babies. See Bremner 1994; Meltzoff 1993; Clifton 1992; Kellman 1995.

34. The distinction of particular and specific identity is mine and is as follows: Particular identity answers to *that*, *which*, *where*, or *when*. Specific identity answers to *what*. Every existent consists of both a particular and a specific identity (Boydstun 1991, 43–46; 1995, 110).

35. The sense of *implicit* here is extracted from the relevant cognitive-development research literature (viz., Gelman and Meck 1983, 344). The child is said to have implicit knowledge of the counting principles if she engages in behavior that is systematically governed by those principles, even though she cannot state them. (See Note 40 for the principles.) Gelman and Meck liken this implicitness of the counting principles at this stage of cognitive development to the way in which we are able to conform to certain rules of syntax when speaking correctly without being able to state those rules. That much seems right, but there is a further distinction I want to make. The child's implicit counting principles are being learned (and taught) as an integral part of learning to properly count aggregations explicitly, expressly. In contrast, we can (or anyway, my preliterate Choctaw ancestors centuries past could) live out our lives, speaking fine in our mother tongue, following right rules of syntax, yet without being able to state those rules; indeed, without even knowing any of the terminology of syntax. Our learning of tacit rules of syntax is not for the sake of becoming able to follow them

explicitly, only tacitly.

In the present developmental discussion, I shall reserve the term *implicit* to indicate that an operative rule is not only tacit, but has become operative as an integral part of becoming explicitly operative. The tacit logical principles, whose acquisition according to Macnamara is traced in the text, are not implicit in my present sense.

There is, of course, another sense of *implicit* that I am also happy to use. That is the logicomathematical sense, which was pertinent to our analysis section. It is in that sense that we say a certain theorem is implicit in a set of axioms; Hertz' wave equation for propagation of electromagnetic radiation is implicit in Maxwell's field equations; an inverse-cube central force law is implicit in a spiral orbit; dimension reductions are implicit in Kolmogorov superposition-based neural networks; certain measure relations are implicit in any similarity discerned in perception; or certain measure relations are implicit in a concept class. *Cf.* Rand 1969–71, 159–62, 178–79; Campbell 2002, 294–96, 300–10; Boydstun 1996, 201–2.

36. Drawn out into our adult expression, here is the logic tacitly put to work by the toddler at this stage: There is a unique kind (class) of which Star is a member, and any object is a ball if and only if it is a member of that kind. For any particular ball, there is a unique member of the kind *ball*, and as long as that member exists, it is identical (totally same) with that particular ball (Macnamara 1986, 137–39). I should say that such working interpretive principles render one's perceptual knowledge conceptual. One has conceptual knowledge even at the single-words stage of language development.

My example of proper naming of a special ball Star is contrived for convenience of illustrating the tacit logical resource. Toddlers at this stage are likely to restrict proper names to particular (real or make-believe) animate entities possessing mentality (Bloom 2000, 130–31).

37. By 24 months, the child is using two-word utterances such as "Mommy sit!" and "guy there" and "I know [how to do it]" (Bremner 1994, 252–53; Nelson 1996, 112, 124–25). Up to about this time, when grammar begins to develop, "words learned remain tied to their world models and do not form systems of their own" (Nelson 1996, 128). In terms of Deacon's iconic, indexical, and symbolic levels of representation (1997, 70–83), I should say that concepts at the single-words stage are indexical representations, and these concepts will become symbolic representations with the onset of grammar. Rand's conceptual level of representation cuts across Deacon's indexical and symbolic levels.

All three levels of representational cognition—even the iconic level (e.g., drawing a stick man)—are active, deliberate, and constructive. I take the membership relation, which is essential for concepts, classes, and sets, to require this sort of active generation, from our first concept to our last. In this way, the membership relation is unlike perceptual relations of similarity, proximity, or containment (*cf.* Rand [1961] 1964, 20; Maddy 1997, 90–94, 108–9, 152 n. 30, 172–76, 185–88).

38. See further Boydstun 1990, 16–18; Minsky [1974] 1997, 111–17; Johnson 1987, 23–30, 102–4; Iverson and Thelen 1999; and Nelson 1996, 16–17.

39. *Cf.* Kelley and Krueger 1984, 47, 52. In saying that this tacit logical principle of application is a surrogate for a concept's definition, I mean to say only that the tacit principle accomplishes the main function that an explicit definition accomplishes. I do not mean to say that the tacit principle is additionally an implicit definition in the developmental sense of *implicit* (as in Note 35). Macnamara's tacit logical principle of application is needed just as much for

concepts of things in terms of merely characteristic features as it is for concepts of things in terms of defining features (*cf.* Bloom 2000, 18–19).

During the first few years of speech, we evidently tend to conceive of things in terms of characteristic features. After about age 5, there is a developmental shift to conceiving of things in terms of defining features. The course of this shift, which occurs at different times in different domains of knowledge, has been partially charted by Frank Keil 1989; see also Boydstun 1990, 34–37. The shift need never occur for all our concepts. [In a preliterate culture (my Choctaw ancestors again), is the shift so extensive as in our culture? See Olson 1994.] Acquiring a tacit logical principle of application is not for the sake of becoming able to conceive of things in terms of defining features.

40. The child has gone far beyond learning first words (roughly months 12 to 18) by the time she is learning to count. By 30 months, the basic linguistic system has become established and is fairly stable (Nelson 1996, 106). Not until around 36 months or beyond does the child have an implicit grasp of the elementary principles of counting: assign one-label-for-one-item, keep stable the order of number labels recited, assign final recited number as the number of items in the counted collection, realize that any sort of items can be counted, and realize that the order in which the items are counted is irrelevant (Gelman and Meck 1983; Butterworth 1999, 109–16).

At 22 months, a child in my family could "say his numbers." This competence is not essentially different from being able to "say his ABC's" (Bloom 2000, 215). Rand may have mistaken the onset of recitation of count-word sequences with onset of ability to count.

41. *Cf.* Macnamara 1986, 143; Burgess 1998, 10–11; Boolos 1984, 72.

42. For Ockham on comparative similarity, see Maurer 1994, 387, 389. For more on comparative difference and comparative similarity in the theory of concept formation, especially in Rand's theory, see Kelley and Krueger 1984, 52–61, and Kelley 1984, 336–45. See also Jetton 1998, 63–72, and Livingston 1998, 15–21.

43. *Cf.* Armstrong 1997, 64–65, for a related extravagance, which he boldly embraces. The extravagant implication I pose is avoided by me in one way, for another way, consider Jetton 1991.

44. See Quine 1969, 117–23; Krantz, Luce, Suppes, and Tversky 1989, 207–22; Nosofsky 1992, 38–40.

45. The General Relativity principle that freely falling bodies follow time-like geodesics of space-time is subject to analytical challenges (Torretti [1983] 1996, 176–81) and to empirical tests, such as whether Earth and Moon have different accelerations towards the sun (Ciufolini and Wheeler 1995, 14, 88, 113–15). Contrast those methods of evaluating conjectures in natural science with the methods of evaluating various candidate axioms for a formal discipline such as set theory (Maddy 1997). We should expect the forms of evaluation appropriate to measurement conjectures for a theory of concepts and concept classes to lie between forms appropriate to natural science and forms appropriate to the formal disciplines of mathematics, set theory, and logic.

46. This essay was studied at the 2003 Advanced Summer Seminar of The Objectivist Center. The significance of the present work was clearly appreciated. The session indicated that reference to an accessible general overview of modern mathematics would be helpful. I heartily recommend MacLane 1986.

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